Let $f : [0,1] \to (0,\infty)$ be a continuous function. Find the value of

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{f(\frac{1}{n})}+\sqrt[n]{f(\frac{2}{n})}+\cdots+\sqrt[n]{f(\frac{n}{n})}}{n}\right)^n.$$

Solutions

- **5164:** *Proposed by Kenneth Korbin, New York, NY*

A triangle has integer length sides (a,b,c) such that $a-b=b-c$. Find the dimensions of the triangle if the inradius $r=\sqrt{13}$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

If a, b and c are the side lengths of the triangle then the inradius r is given by the formula

$r=\dfrac{1}{2}\sqrt{\dfrac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}}$. (see, e.g., http//mathworld.wolfram.com/Inradius.html).

By assumption, $c=2b-a$. So

$$\sqrt{13}=\frac{1}{2}\sqrt{\frac{(3b-2a)(2a-b)}{3}}, \text{ or equivalently}$$

$$(3b-2a)(2a-b)=156.$$

Obviously b is even. (If b were odd, then both $3b-2a$ and $2a-b$ are odd, and therefore their product would be odd, which is not true.) So $b=2b'$ and this gives the equation

$$(3b'-a)(a-b')=39.$$

Note that $39=xy$ is the product of two integers. So,

$$(x,y)\in\{(1,39),(3,13),(13,3),(39,1),(-1,-39),(-3,-13),(-13,-3),(-39,-1)\}.$$

If $3b'-a=x$ and $a-b'=y$, then

$$\begin{aligned} b' &= \frac{x+y}{2}, \text{ and}\\ a &= \frac{x+3y}{2}. \end{aligned}$$

We find $(a,b,c)\in\{(59,40,21),(21,16,11),(11,16,21),(21,40,59)\}$, and we easily verify that each triplet satisfies the triangle inequality.

Solution 2 by Arkady Alt, San Jose, CA

Let F and s be the area and semiperimeter. Since $a+c=2b$ then $s=\dfrac{a+b+c}{2}=\dfrac{3b}{2}$, and using $F=\sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)}=sr$ we obtain

$$\left(s-a\right)\left(s-b\right)\left(s-c\right)=sr^2 \iff \left(\frac{3b}{2}-a\right)\left(\frac{3b}{2}-b\right)\left(\frac{3b}{2}-c\right)=13\cdot\frac{3b}{2}$$

$$\iff \quad \left(\frac{3b}{2}-a\right)\left(\frac{3b}{2}-c\right)=39$$

$$\iff \quad \left(\frac{9b^2}{4}-\left(a+c\right)\frac{3b}{2}+ac\right)=39 \iff \left(\frac{9b^2}{4}-2b\cdot\frac{3b}{2}+ac\right)=39$$

$$\iff \quad 4ac-3b^2=12\cdot 13.$$

Thus we have

$$\begin{cases} a+c=2b \\ 4ac-3b^2=156 \end{cases} \implies \begin{cases} 4a\left(2b-a\right)-3b^2=156 \\ c=2b-a \end{cases} \quad \text{if, and only if,}$$

$$\begin{cases} 4a\left(2b-a\right)-3b^2=156 \\ c=2b-a \end{cases} \iff \begin{cases} 8ab-a^2-3b^2=156 \\ c=2b-a. \end{cases}$$

Since $8ab-a^2-3b^2=\left(3b-2a\right)\left(2a-b\right)$ and

$$\begin{cases} a<s \\ b<s \\ c<s \end{cases} \iff \begin{cases} 2a<3b \\ c<s \end{cases} \iff \begin{cases} 2a<3b \\ 2\left(2b-a\right)<3b \end{cases} \iff b<2a<3b$$

then the problem is equivalent to the system

$$(1) \qquad \begin{cases} \left(3b-2a\right)\left(2a-b\right)=156 \\ b<2a<3b. \end{cases}$$

Since $3b-2a\equiv 2a-b \pmod 2$ and $156=2^2\cdot 3\cdot 13=2\cdot 78=6\cdot 26$ then (1) in positive integers is equivalent to

$$\begin{cases} 3b-2a=k \\ 2a-b=m \end{cases} \iff \begin{cases} 2b=k+m \\ 4a=k+3m \end{cases} \iff \begin{cases} a=\dfrac{k+3m}{4} \\ b=\dfrac{k+m}{2} \end{cases},$$

where $\left(k,m\right)\in\left\{\left(2,78\right),\left(78,2\right),\left(6,26\right),\left(26,6\right)\right\}.$

Noting that the inequality $b<2a<3b \iff \dfrac{k+m}{2}<\dfrac{k+3m}{2}<\dfrac{3\left(k+m\right)}{2}$ holds for any positive k,m we finally obtain

$$\left(a,b\right)\in\left\{\left(59,40\right),\left(21,40\right),\left(21,16\right),\left(11,16\right)\right\}.$$

Thus, $(a, b, c) \in \{(59, 40, 21), (21, 40, 59), (21, 16, 11), (11, 16, 21)\}$ are all solutions of the problem.

Comment by David Stone and John Hawkins, Statesboro, GA. In their featured solutions to SSM 5146 (May 2011 issue) both Kee-Wai Lau and Brian Beasley found all integral triangles with in-radius $\sqrt{13}$. Note that the condition $a - b = b - c$ is equivalent to $b = (a + c)/2$. That is, irrespective of how one might label or order the sides, the side b must be the "middle-length" side , the average of the other two sides.

Also solved by Brain D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo TX; Bruno Salgueiro Fanego, Viveiro, Spain; Tania Moreno García, University of Holguín (UHO), Holguín, Cuba jointly with José Pablo Suárez Rivero, University of Las Palmas de Gran Canaria (ULPGC), Spain; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, University of Technology, Sydney, Australia jointly with Elton Bojaxhiu, Kriftel, Germany; Sugie Lee, John Patton, and Matthew Fox (jointly; students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Jim Wilson, Athens, GA, and the proposer.

- **5165:** *Proposed by Thomas Moore, Bridgewater, MA*

"Dedicated to Dr. Thomas Koshy, friend, colleague and fellow Fibonacci enthusiast."

Let $\sigma(n)$ denote the sum of all the different divisors of the positive integer n. Then n is perfect, deficient, or abundant according as $\sigma(n) = 2n, \sigma(n) < 2n$, or $\sigma(n) > 2n$. For example, 1 and all primes are deficient; 6 is perfect, and 12 is abundant. Find infinitely many integers that are not the product of two deficient numbers.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \ldots$ be the sequence of primes. We show that for any positive integer n, the integer $\prod_{k=1}^{n+10} p_k$ is not the product of two deficient numbers.

Suppose , on the contrary, that $\prod_{k=1}^{n+10} p_k = ab$, where both a and b are deficient numbers. Clearly a and b are relatively prime and so

$$4\left(\prod_{k=1}^{n+10} p_i\right) = 4ab > \sigma(a)\sigma(b) = \sigma(ab) = \sigma\left(\prod_{k=1}^{n+10} p_k\right) = \prod_{k=1}^{n+10} (1 + p_k).$$

Hence,

$$4 > \prod_{k=1}^{n+10}\left(1 + \frac{1}{p_k}\right) \geq \prod_{k=1}^{11}\left(1 + \frac{1}{p_k}\right) = \frac{3822059520}{955049953} = 4.0019\ldots,$$

which is a contradiction. This completes the solution.

Solution 2 by Stephen Chou, Talbot Knighton, and Tom Peller (students at Taylor University), Upland, IN

All negative numbers have the same numerical divisors as their positive counterparts;